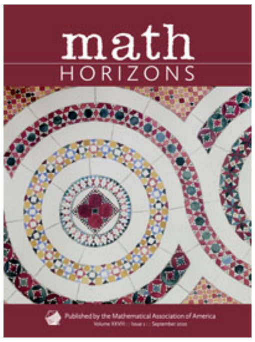

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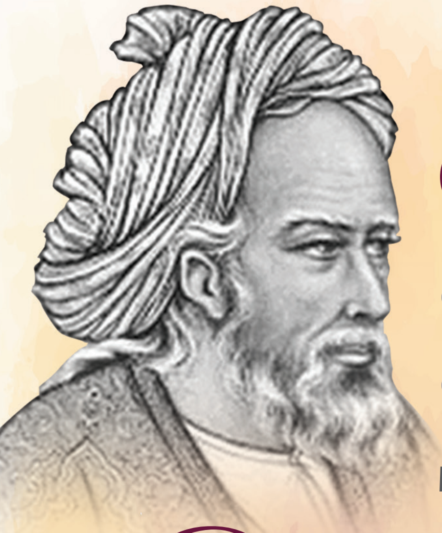


Omar Khayyam: Geometric Algebra and Cubic Equations

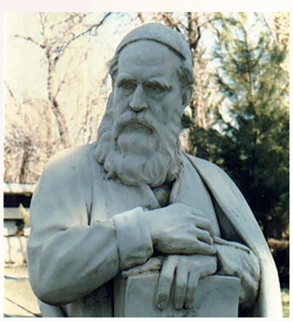
M. Vali Siadat and Alana Tholen

Omar Khayyam ("Khayyam" meaning the tent maker) was born on May 18, 1048, in Nishapur, a city in the state of Khorasan in northeastern Iran, and died on December 4, 1131. Khayyam, while widely known in the Western world as a poet, made significant mathematical contributions.

In his most well-known mathematics publication, "Treatise on Demonstration of Problems of Algebra," Khayyam presented methods to find solutions of cubic equations using geometric constructions by identifying the intersection of hyperbolas, parabolas, circles, and semicircles. In particular, Khayyam classified cubic equations into various types by determining which conics would be used in each geometric construction; only two conics were ever used to solve a single cubic equation (Ing, "The comparison between the methods of solutions for cubic equations," *Math. Medley*, 30 [2003] 90–101).

Khayyam's geometric constructions did not allow for negative or imaginary solutions, as those sets of numbers were not yet discovered (*Omar Khayyam as a Mathematician*, Omar Khayyam Club of America: Rosemary Press, 1919). However, there is some evidence suggesting that Khayyam recognized that certain cubic equations would have more than one positive real solution, while other cubic equations would have no positive real solutions (Amir-Moéz, "Khayyam's solution of cubic equations," *Math. Mag.*, 35 [1962] 269–271).

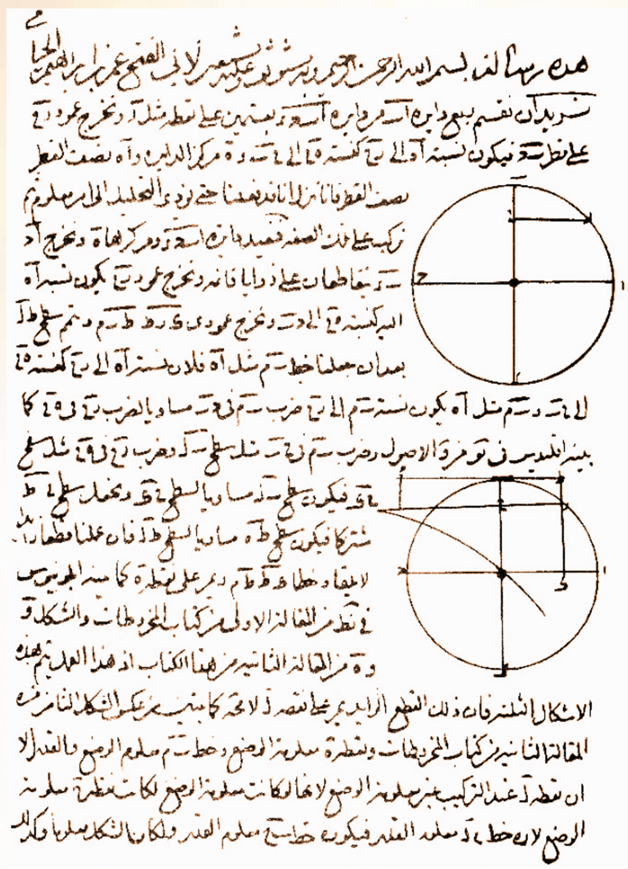
Figure 1. A statue of Omar Khayyam in Iran.



In this respect, we may regard Khayyam as a pioneer in connecting algebra with geometry well before the French mathematician Rene Descartes (1596–1650) was credited with inventing analytic geometry. In Khayyam's Treatise, he refers to an unknown in an algebraic equation as a thing "shay" (in Persian/Arabic).

Medieval Islamic algebra was more conceptual in nature, so that a "xay" or an "x" represented the side length of a square, an x^2 represented the area of a square, an x^3

Figure 2. Cubic equation and intersection of conics; the first page of two-chaptered manuscript kept in Tehran University.



represented the volume of a cube, and an x with a larger degree exponent was represented as the sum of parallelepiped volumes (Kent & Muraki, "A geometric solution of a cubic by Omar Khayyam," *Amer. Math. Monthly*, **123** [2015] 149–160). For example, Khayyam would have translated the equation $x^3 + 4x = x^2 + 16$ as, "Cube and sides equals square and number," and the equation $x^3 + x^2 + 1 = 10x$ as, "Cube, square, and number equals sides."

Khayyam did not attempt to collect all terms on the left side of the equation, which is what we engrain in the minds of mathematics students today due to modern solving methods (graphing and factoring). Instead, Khayyam used the initial setup of the equation to classify the cubic and to dictate which conics he would use in his geometric construction. Khayyam identified 19 various forms of cubic equations, listed in table 1, and he recognized that several could be solved by either taking the cube root of a number or by reducing the cubic to a quadratic equation. For other forms of cubic equations, Khayyam stated that geometric constructions were necessary, and thus, conics must be used.

Geometric Solutions of Cubic Equations

Consider cubic equations in the form $x^3 + bx = c$. Khayyam would have assumed b and c are both positive integers; note that Medieval Islamic mathematicians had no reason to consider the case $c = 0$ because cubic equations related to volumes—if the volume of the cube equaled zero, there would be no cube to consider.

By table 1, the solution of this type of cubic results from the intersection of a parabola and semicircle. The required semicircle has radius $r = c / 2b$ and is centered at $(r, 0)$, while the parabola is given by the equation $y = x^2 / \sqrt{b}$. These conics intersect at $(0, 0)$ as well as one additional point, which we call (x_1, y_1).

Because (x_1, y_1) lies on the parabola, we know that $\sqrt{b}y_1 = x_1^2$, so that $\sqrt{b} / x_1 = x_1 / y_1$. As (x_1, y_1) also lies on the circle centered at $(r, 0)$ with radius r, we conclude that $(x_1 - r)^2 + y_1^2 = r^2$ or that $y_1^2 = -x_1^2 + 2x_1 r$. From this, we deduce that $x_1 / y_1 = y_1 / (2r - x_1)$.

We thus have $\sqrt{b} / x_1 = x_1 / y_1 = y_1 / (2r - x_1)$. But then

$$\frac{b}{x_1^2} = \frac{x_1}{y_1} \cdot \frac{y_1}{2r - x_1} = \frac{x_1}{2r - x_1},$$

Table 1. Classification of all 19 types of Khayyam's cubic equations along with solving methods (see Ing, 2003 and Rizvi, "Umar Khayyam as a geometrician: A survey," *Islamic Studies*, **24** [1985] 193–204).

Equation form	Translation of equation	Method of solution
$x^3 = c$	Cube equals number	Cube root c
$x^3 = bx$	Cube equals sides	Reduce to $x^2 = b$
$x^3 = ax^2$	Cube equals square	Reduce to $x = a$
$x^3 + ax^2 = bx$	Cube and square equal sides	Reduce to $x^2 + ax = b$
$x^3 = ax^2 + bx$	Cube equals square and sides	Reduce to $x^2 = ax + b$
$x^3 + bx = ax^2$	Cube and sides equal square	Reduce to $x^2 + b = ax$
$x^3 + bx = c$	Cube and sides equal number	Parabola and semicircle
$x^3 + c = bx$	Cube and number equal sides	Parabola and hyperbola
$x^3 + c = ax^2$	Cube and number equal square	Parabola and hyperbola
$x^3 + ax^2 = c$	Cube and square equal number	Parabola and hyperbola
$x^3 = ax^2 + c$	Cube equals square and number	Parabola and hyperbola
$x^3 = bx + c$	Cube equals sides and number	Parabola and hyperbola
$x^3 + ax^2 = bx + c$	Cube and square equal sides and number	Hyperbola and hyperbola
$x^3 + c = ax^2 + bx$	Cube and number equal square and sides	Hyperbola and hyperbola
$x^3 = ax^2 + bx + c$	Cube equals square, sides and number	Hyperbola and hyperbola
$x^3 + ax^2 + c = bx$	Cube, square and number equal sides	Hyperbola and semicircle
$x^3 + ax^2 + bx = c$	Cube, square, and sides equal number	Hyperbola and semicircle
$x^3 + bx = ax^2 + c$	Cube and sides equal square and number	Hyperbola and circle
$x^3 + bx + c = ax^2$	Cube, sides, and number equal square	Hyperbola and circle

and as such $x_1^3 = 2rb - bx_1 = c - bx_1$. Hence, x_1 is indeed a solution to the original cubic equation.

To see the method in action, we'll find the solution of the equation $x^3 + 4x = 5$. We construct a semicircle with a radius of 5/8 units centered at (5/8, 0) and the parabola $y = x^2 / 2$ as shown in figure 3. These conics intersect at the point (1,1/2) so that $x = 1$ is the solution to the cubic equation $x^3 + 4x = 5$. This method also led Khayyam to the generalization that whenever $c = b + 1$, the value $x = 1$ is a solution of the equation $x^3 + bx = c$.

Other cubic equations that have clear solutions via graphing include the following: $x^3 + 4x = 16$, $x^3 + 3x = 36$, and $x^3 + 20x = 225$. We encourage the reader to find constructions revealing the solutions $x = 2$, $x = 3$, and $x = 5$, respectively.

The previous equations yield positive integer solutions, but sometimes cubic equations have positive non-integer solutions. When this occurs, the point of intersection of the conics is not clear from the geometric construction. Therefore, we must consider the solution differently. To illustrate, we perform the geometric construction for $x^3 + 9x = 36$ in figure 4. We can see that the solution lies between $x = 2$ and $x = 3$, and it appears close to $x = 2.5$. Using modern technology, we find that $x \approx 2.4218$, which is close to our estimation.

Khayyam did not have access to such technology, so instead he represented the solution to this cubic equation as a line segment with a length from the origin to the perpendicular bisector of the x-axis passing through the intersection point. In fact, Khayyam represented solutions to cubic equations as lengths whenever the graphic solution was not clear. Because he kept his constructions standardized, Khayyam could compare the lengths of multiple solutions to different cubic equations and construct generalizations about cubic equations and the solutions they yielded (Kent & Muraki, 2015).

While Khayyam only used a semicircle and parabola to solve $x^3 + bx = c$, he used related techniques when working with parabolas and hyperbolas, semicircles and hyperbolas, etc. We invite the reader to give modern proofs, similar to the one we have provided, showing that the cubic equations listed in table 1 have solutions arising from the intersection of the stated conics.

In his treatise, Khayyam writes "Whoever thinks algebra is a trick in obtaining unknowns has thought it in vain; no attention should be paid to the fact that algebra and geometry are different in appearance." In short, Khayyam considered algebra and geometry simultaneously, allowing him to study and solve cubic equations using geometric constructions. He did this well before the 16th century when del Ferro, Tartaglia, and Cardano discovered the modern solution of the cubic.

Khayyam's work is just one example of historical mathematics that is often overlooked by modern undergraduate mathematics programs. Not enough students have the opportunity to practice ancient mathematical techniques, nor are students able to learn about the developmental process of specific topics. Outside of a historical mathematics class, students may never be exposed to this sector of collegiate mathematics.

Other Contributions

Among other things, Khayyam also contributed to the understanding of Euclid's parallel postulate and explained the construction of binomial coefficients as a triangular array of numbers (now known as Pascal's triangle) in his treatise. Moreover, he made several non-mathematical contributions to the academic world; Khayyam was also an ancient philosopher, a poet, and an astronomer.

In 1074, the Saljuk king, Sultan Malik Shah, commissioned Khayyam to build an observatory, with the assistance of other prominent scientists, in the city of Isfahan. Their mission was to reform the Persian calendar. The new Jalali calendar, named after the Sultan, measured the length of the solar year as 365.24219858156 days and has been the basis of the Iranian calendar since the 11th century. The

Figure 3. Geometric solution of $x^3 + 4x = 5$ using a parabola (purple) and a semicircle (green).

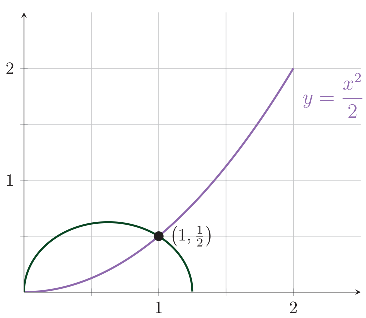


Figure 4. Geometric solution of $x^3 + 9x = 36$ as the intersection of the parabola $y = x^2 / 3$ (in blue) and the circle of radius 2 centered at (2,0) (in orange). The length of red segment from the origin to x_1 represents the solution as Khayyam did.

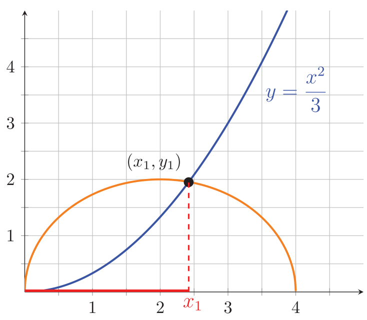


Figure 5. Tomb of Omar Khayyam in Nishapur, Iran.

Iranian new year, Norooz, begins within a day of March 21 and is celebrated in Afghanistan, Tajikistan, Uzbekistan, Kyrgyzstan, Azerbaijan, as well as Iran.

Khayyam's philosophical thoughts could be understood from his writings on the subject in the form of six philosophical papers attributed to him. In these, he discusses the relationship between existence and universals, free will and determinism, being and necessity, transcendence in existence, knowledge of universality of existence, and natural phenomena.

His philosophy can also be more clearly understood from his poems, the four-line verses called Rubaiyat (quatrains). For example, Khayyam states that, "The world didn't benefit from me coming into it, nor did it benefit from me leaving it. And no one could ever tell me the purpose of this coming and going." Khayyam also writes that, "Beyond the earth, beyond the farthest skies, I try to find Heaven and Hell; then I hear a solemn voice that says, 'Heaven and Hell are inside.'"

Khayyam goes on, in his enigmatic message, with a poem that encourages happiness and enjoyment of life: "If with wine you are drunk, be happy. If seated with a beautiful, be happy. Since the end purpose of the universe is nothingness, hence picture your nothingness, then while you are, be happy." In another poem referring to vanity in people's lives, he says, "Those who have gone forth, thou cup-bearer; Have fallen upon the dust of pride, thou cup-bearer; Drink wine and hear from me the truth: Hot air is all that they have said, thou cup-bearer."

Exposing the hypocrisy of the religious authority, he writes, "O Mullah, we do much more work than you; Even when we are drunk, we are still more sober than you; You drink people's blood and we drink the grape's blood [wine]; Let's be fair, which one of us is more immoral?"

Finally, displaying the dynamics of human life as one of accomplishment, despite coming from dust and disappearing into dust, he beautifully writes, "Myself when young did eagerly frequent doctor and saint, and heard great argument about it and about: but evermore came out by the same door as I went in."

While it is not possible to accurately describe Khayyam's philosophical thought, it can be said that he was neither ascetic nor materialistic. Khayyam believed in neither mysticism nor spiritualism. Although from Islamic background, he did not formally subscribe to any religion, but rather he leaned toward agnostic skepticism. He was a naturalist, believing in the power of knowledge and wisdom as means to the salvation of human beings. He advocated happiness for all human beings in their present life, not in the "afterlife."

As a free-thinking scientist, Khayyam used logic and reason in pursuit of truth. He questioned the existing order and critiqued whatever that was considered normal. He defied authority and challenged the established norms. Omar Khayyam was truly an iconoclast not only in mathematics, but also in many facets of Medieval Islamic life. Let his work live on, and let us continue studying his geometric approach to solving cubic equations.

M. Vali Siadat, *PhD, DA, is a distinguished professor of mathematics at City Colleges of Chicago (Richard J. Daley College). His research area is in mathematics and mathematics education. He is the winner of the AMS "2019 Award for Impact on the Teaching and Learning of Mathematics," and the MAA "2009 Deborah and Franklin Tepper Haimo Award for Distinguished Teaching of Mathematics," among others. He is a co-developer of the Keystone model of teaching and learning of mathematics. He enjoys playing volleyball and jogging, and he is a movie buff who enjoys classical movies.*

Alana Tholen, *PhD candidate at the University of Iowa, is finishing her degree in Mathematics Education with cognates in Gifted Education and Historical Mathematics. She was first introduced to* Math Horizons *during her undergraduate years at Coe College in Cedar Rapids, Iowa; since then, she has been an avid reader of the magazine. She is currently teaching AP Statistics while writing her dissertation, which focuses on increasing equity in the mathematics classroom. Her hobbies include writing and illustrating children's books that promote the pursuit of STEM.*

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